Automated Complexity Analysis of Rewrite Systems

Florian Frohn

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December 11, 2018

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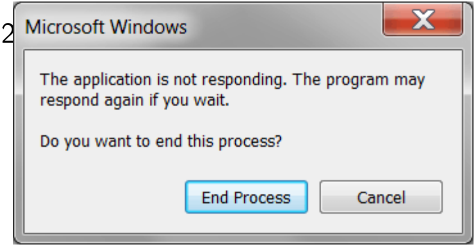
December 11, 2



What is Complexity Analysis?

input: $x, y \in \mathbb{N}$
$z = 0$
while $x > 0$ **do**
  $x = x - 1$
  $z = z + y$
end
return z

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- x loop iterations

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- 5 operations per iteration

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- upper bound: $7 \cdot x + 4$, ...

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- x loop iterations
- 5 operations per iteration
- complexity $5 \cdot x + 2$
- upper bound: $7 \cdot x + 4$, ...
- lower bound: $x + 1$, ...

Goal: compute such bounds automatically

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Complexity Analysis with AProVE

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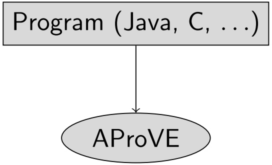

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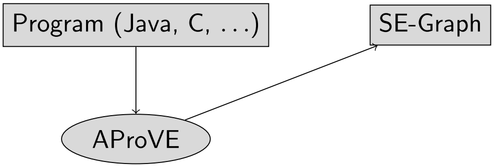

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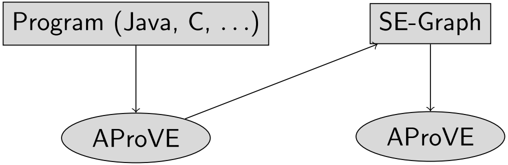

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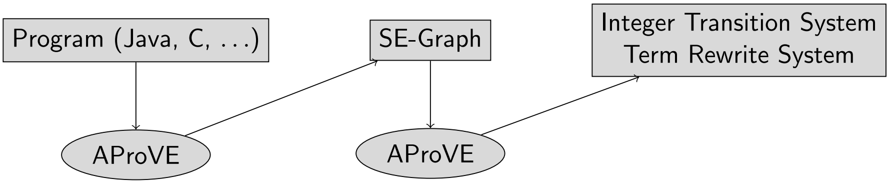

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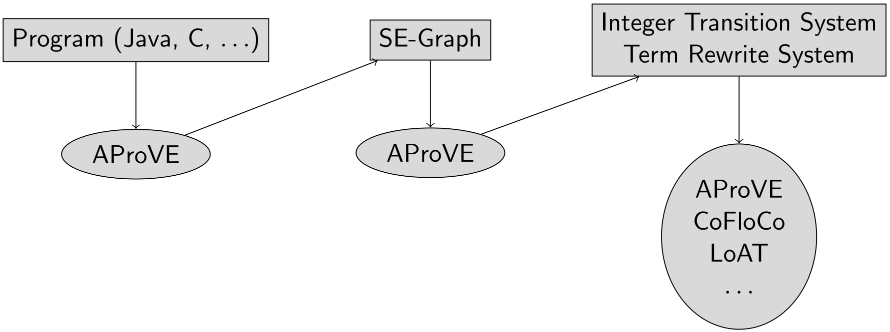

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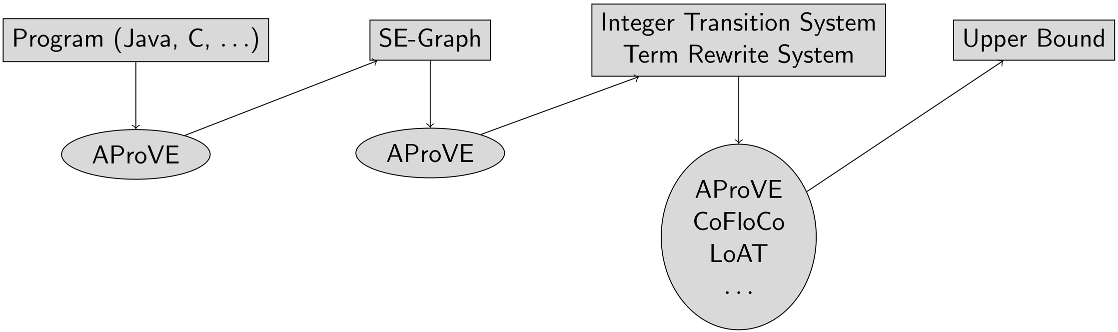

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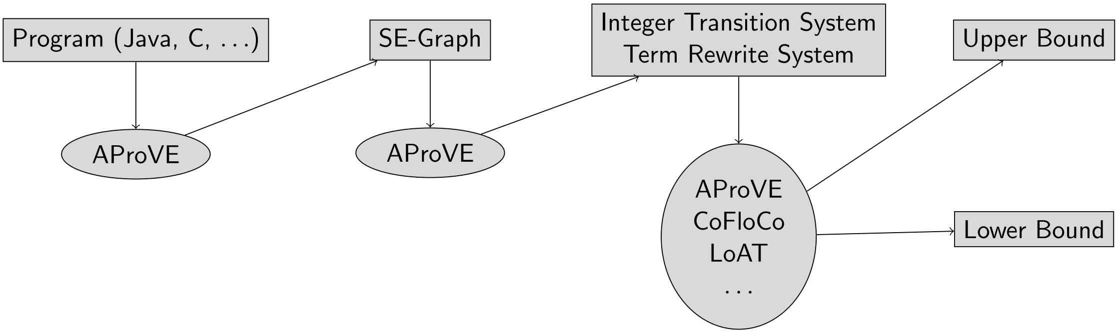

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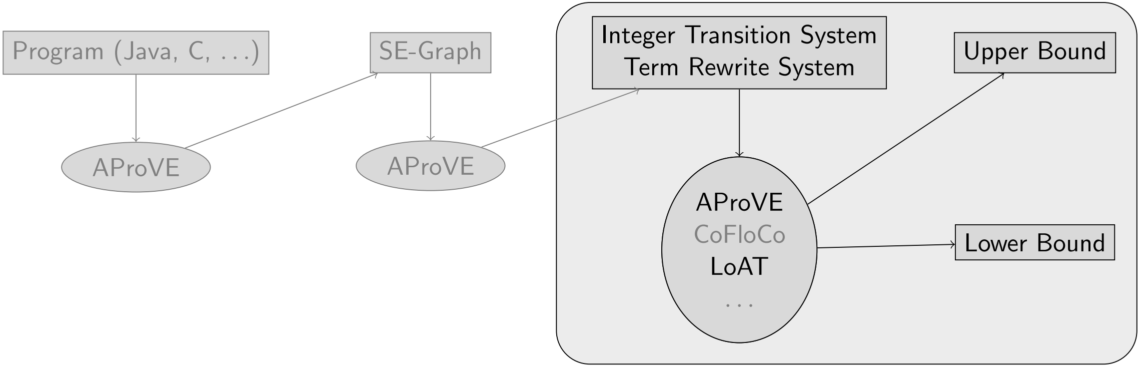

Rule-Based Representation of Programs

Rule-Based Representation of Programs

Example (Integer Transition System (ITS))

$$\begin{array}{llll} \mathtt{init}(x,y) & \xrightarrow{1} & \mathtt{while}(x,y,0) & [x \geq 0 \wedge y \geq 0] \\ \mathtt{while}(x,y,z) & \xrightarrow{5} & \mathtt{while}(x-1,y,z+y) & [x > 0] \\ \mathtt{while}(x,y,z) & \xrightarrow{1} & \mathtt{return}(z) & [x = 0] \end{array}$$

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Example (Term Rewrite System (TRS))

$$
\begin{array}{lll}
\texttt{length}(\mathbf{cons}(x,x')) & \to & \mathsf{s}(\texttt{length}(x')) \\
\texttt{length}(\mathbf{nil}) & \to & \mathbf{0}
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$$

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$$\begin{array}{lll} 0 & \curvearrowright & \mathbf{0} \\ 1 & \curvearrowright & \mathsf{s}(\mathbf{0}) \\ 2 & \curvearrowright & \mathsf{s}(\mathsf{s}(\mathbf{0})) \end{array}$$

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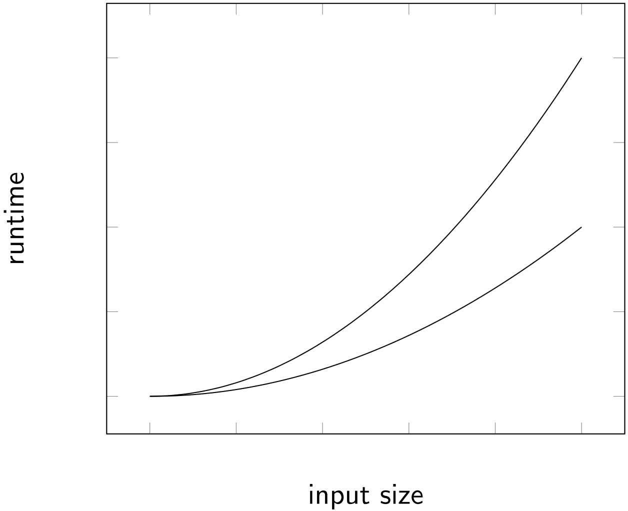
Upper, Lower, Worst Case, Best Case, ...



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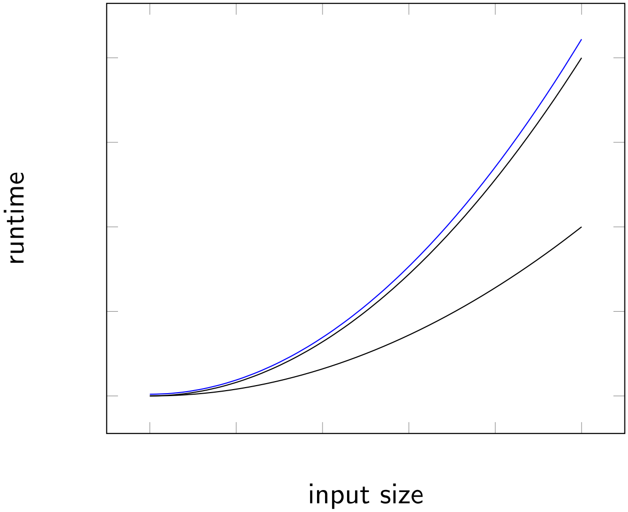

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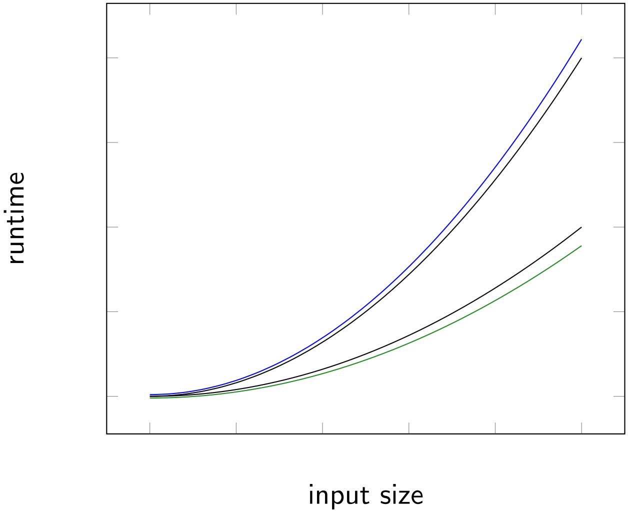

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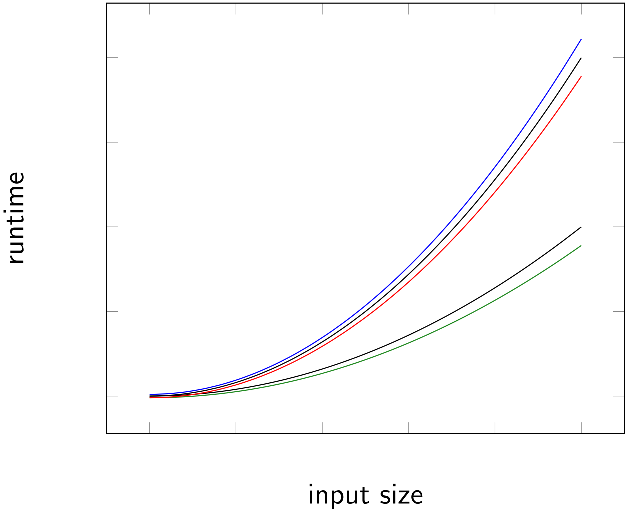

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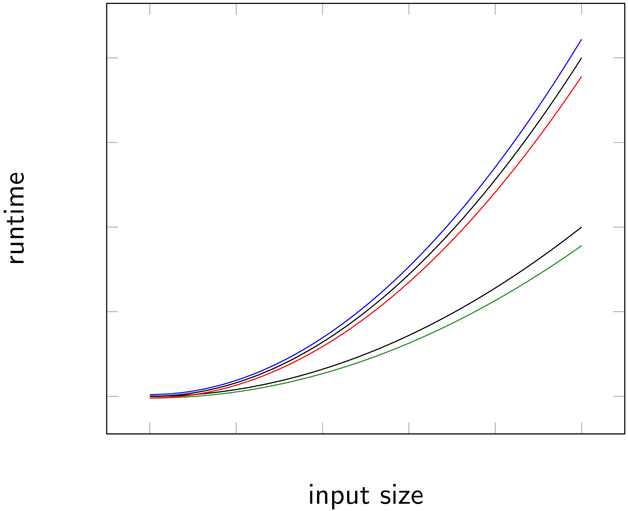

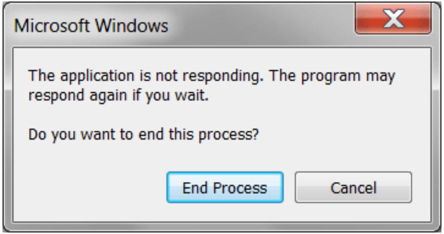

Outline

1. Introduction
2. **Lower Bounds for ITSs**
3. Lower Bounds for TRSs
4. Constant Upper Bounds for TRSs

Evaluating ITSs

Example (ITS)

$$\begin{array}{lcll} \texttt{while}(x, y, z) & \xrightarrow{5} & \texttt{while}(x-1, y, z+y) & [x > 0] \\ \texttt{while}(x, y, z) & \xrightarrow{1} & \texttt{return}(z) & [x = 0] \end{array}$$

Evaluating ITSs

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$$\texttt{while}(x, y, z) \xrightarrow{5} \texttt{while}(x-1, y, z+y) \quad [x > 0]$$
$$\texttt{while}(x, y, z) \xrightarrow{1} \texttt{return}(z) \quad [x = 0]$$

Example (Evaluation)

$\texttt{while}(2, 2, 0)$

Evaluating ITSs

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Example (Evaluation)

$\texttt{while}(2, 2, 0)$	$[x/2, y/2, z/0]$

Evaluating ITSs

Example (ITS)

$$\begin{array}{llll} \texttt{while}(x,y,z) & \xrightarrow{5} & \texttt{while}(x-1,y,z+y) & [x>0] \\ \texttt{while}(x,y,z) & \xrightarrow{1} & \texttt{return}(z) & [x=0] \end{array}$$

Example (Evaluation)

$$\texttt{while}(2,2,0) \quad \Big| \quad [x/2, y/2, z/0] \models x > 0$$

Evaluating ITSs

Example (ITS)

$$\begin{array}{llll} \texttt{while}(x,y,z) & \xrightarrow{5} & \texttt{while}(x-1,y,z+y) & [x>0] \\ \texttt{while}(x,y,z) & \xrightarrow{1} & \texttt{return}(z) & [x=0] \end{array}$$

Example (Evaluation)

$$\begin{array}{rl|l} & \texttt{while}(2,2,0) & [x/2,y/2,z/0] \models x>0 \\ \xrightarrow{5} & \texttt{while}(1,2,2) & \end{array}$$

Evaluating ITSs

Example (ITS)

$$\texttt{while}(x, y, z) \xrightarrow{5} \texttt{while}(x-1, y, z+y) \quad [x > 0]$$
$$\texttt{while}(x, y, z) \xrightarrow{1} \texttt{return}(z) \quad [x = 0]$$

Example (Evaluation)

	$\texttt{while}(2, 2, 0)$	$[x/2, y/2, z/0] \models x > 0$
$\xrightarrow{5}$	$\texttt{while}(1, 2, 2)$	$[x/1, y/2, z/2]$

Evaluating ITSs

Example (ITS)

$$\texttt{while}(x, y, z) \xrightarrow{5} \texttt{while}(x-1, y, z+y) \quad [x > 0]$$
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	$\texttt{while}(2, 2, 0)$	$[x/2, y/2, z/0] \models x > 0$
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$$\begin{array}{ll|l} & \texttt{while}(2,2,0) & [x/2,y/2,z/0] \models x>0 \\ \xrightarrow{5} & \texttt{while}(1,2,2) & [x/1,y/2,z/2] \models x>0 \\ \xrightarrow{5} & \texttt{while}(0,2,4) & \end{array}$$

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\end{array}
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Example (Evaluation)

$$
\begin{array}{ll|l}
 & \texttt{while}(2, 2, 0) & [x/2, y/2, z/0] \models x > 0 \\
\xrightarrow{5} & \texttt{while}(1, 2, 2) & [x/1, y/2, z/2] \models x > 0 \\
\xrightarrow{5} & \texttt{while}(0, 2, 4) & [x/0, y/2, z/4] \not\models x > 0
\end{array}
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Loop Acceleration

Example (Loop Acceleration)

$$\texttt{while}(x, y, z) \xrightarrow{5} \texttt{while}(x - 1, y, z + y) \qquad [x > 0]$$

Loop Acceleration

Example (Loop Acceleration)

$$\begin{array}{llll} & \mathtt{while}(x,y,z) & \xrightarrow{5} & \mathtt{while}(x-1,y,z+y) & [x>0] \\ \curvearrowright & \mathtt{while}(x,y,z) & \xrightarrow{5\cdot n} & \mathtt{while}(x-n,y,z+y\cdot n) & [x>0 \land 0<n\leq x] \end{array}$$

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- $5 \cdot n$: cost of n iterations
- $x - n$: value of x after n iterations

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Example (Recurrence Equations)

$$\begin{array}{lll} x^{(n)} & = & x^{(n-1)} - 1 \\ y^{(n)} & = & y^{(n-1)} \\ z^{(n)} & = & z^{(n-1)} + y^{(n-1)} \end{array}$$

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\begin{aligned}
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Chaining

Example

$$
\begin{aligned}
\mathtt{init}(x, y) &\xrightarrow{1} \mathtt{while}(x, y, 0) && [x \geq 0 \land y \geq 0] \\
\mathtt{while}(x, y, z) &\xrightarrow{5 \cdot n} \mathtt{while}(x - n, y, z + y \cdot n) && [x > 0 \land 0 < n \leq x] \\
\mathtt{while}(x, y, z) &\xrightarrow{1} \mathtt{return}(z) && [x = 0]
\end{aligned}
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Chaining

Example

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Chaining

Example

$$\begin{array}{llll} \texttt{init}(x,y) & \xrightarrow{1} & \texttt{while}(x,y,0) & [x \geq 0 \land y \geq 0] \\ \texttt{while}(x,y,0) & \xrightarrow{5 \cdot x} & \texttt{while}(0,y,y \cdot x) & [x > 0] \\ \texttt{while}(x,y,z) & \xrightarrow{1} & \texttt{return}(z) & [x = 0] \end{array}$$

$$\curvearrowright \quad \texttt{init}(x,y) \xrightarrow{5 \cdot x+2} \texttt{return}(y \cdot x) \quad [x > 0 \land y \geq 0]$$

Chaining

Example

$$\begin{array}{llll} & \mathtt{init}(x,y) & \xrightarrow{1} & \mathtt{while}(x,y,0) & [x \geq 0 \land y \geq 0] \\ & \mathtt{while}(x,y,0) & \xrightarrow{5\cdot x} & \mathtt{while}(0,y,y\cdot x) & [x > 0] \\ & \mathtt{while}(x,y,z) & \xrightarrow{1} & \mathtt{return}(z) & [x = 0] \\ \curvearrowright & \mathtt{init}(x,y) & \xrightarrow{5\cdot x+2} & \mathtt{return}(y\cdot x) & [x > 0 \land y \geq 0] \end{array}$$

$\curvearrowright$ whole program compressed into a single rule

Chaining

Example

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$$\curvearrowright \quad \texttt{init}(x,y) \xrightarrow{5 \cdot x + 2} \texttt{return}(y \cdot x) \quad [x > 0 \land y \geq 0]$$

- $\curvearrowright$ whole program compressed into a single rule
- $\curvearrowright$ lower bound $5 \cdot x + 2$ for all $x > 0, y \geq 0$

Chaining

Example

$$
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\texttt{init}(x,y) & \xrightarrow{1} & \texttt{while}(x,y,0) & [x \geq 0 \land y \geq 0] \\
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 & & & \\
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\end{array}
$$

$\curvearrowright$ whole program compressed into a single rule

$\curvearrowright$ lower bound $5 \cdot x + 2$ for all $x > 0, y \geq 0$

Works for arbitrary programs!

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first technique to infer worst case lower bounds for ITSs

Lower Bounds for ITSs – Wrap-Up

first technique to infer worst case lower bounds for ITSs

- key idea: *loop acceleration*

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  - ...

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Lower Bounds for ITSs – Wrap-Up

first technique to infer worst case lower bounds for ITSs

- key idea: *loop acceleration*
- many other features, e.g.:
  - non-determinism
  - asymptotic bounds
  - recursive `void` functions
  - SMT encodings
  - ...

	LoAT							
Best Upper Bound: rc(n)	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^4)$	EXP	∞
$\mathcal{O}(1)$	(132)	–	–	–	–	–	–
$\mathcal{O}(n)$	37	126	–	–	–	–	–
$\mathcal{O}(n^2)$	8	14	35	–	–	–	–
$\mathcal{O}(n^3)$	2	–	2	1	–	–	–
$\mathcal{O}(n^4)$	1	–	–	–	2	–	–
EXP	–	–	–	–	–	5	–
∞	53	31	1	–	–	–	176

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Outline

Evaluating TRSs

Example (TRS)

$$\mathtt{inf}(y) \longrightarrow \mathbf{cons}(y, \mathtt{inf}(\mathrm{s}(y)))$$

Evaluating TRSs

Example (TRS)

$$\texttt{inf}(y) \to \mathbf{cons}(y, \texttt{inf}(\mathbf{s}(y)))$$

Example (Evaluation)

$$\texttt{inf}(\mathbf{0})$$

Evaluating TRSs

Example (TRS)

$$\mathtt{inf}(y) \longrightarrow \mathbf{cons}(y, \mathtt{inf}(\mathrm{s}(y)))$$

Example (Evaluation)

$\underline{\mathtt{inf}(\mathbf{0})}$ | $[y/\mathbf{0}]$

Evaluating TRSs

Example (TRS)

$$\mathtt{inf}(y) \to \mathbf{cons}(y, \mathtt{inf}(\mathrm{s}(y)))$$

Example (Evaluation)

$$\underline{\mathtt{inf}(\mathbf{0})} \qquad [y/\mathbf{0}]$$

$$\to \underline{\mathbf{cons}(\mathbf{0}, \mathtt{inf}(\mathrm{s}(\mathbf{0})))}$$

Example (TRS)

$$\mathtt{inf}(y) \to \mathbf{cons}(y, \mathtt{inf}(\mathbf{s}(y)))$$

Example (Evaluation)

	$\mathtt{inf}(\mathbf{0})$	$[y/\mathbf{0}]$
$\to$	$\mathbf{cons}(\mathbf{0}, \underline{\mathtt{inf}(\mathbf{s}(\mathbf{0}))})$	$[y/\mathbf{s}(\mathbf{0})]$

Example (TRS)

$$\texttt{inf}(y) \to \mathbf{cons}(y, \texttt{inf}(\mathsf{s}(y)))$$

Example (Evaluation)

$$
\begin{array}{ll|l}
 & \texttt{inf}(\mathbf{0}) & [y/\mathbf{0}] \\
\to & \mathbf{cons}(\mathbf{0}, \underline{\texttt{inf}(\mathsf{s}(\mathbf{0}))}) & [y/\mathsf{s}(\mathbf{0})] \\
\to & \mathbf{cons}(\mathbf{0}, \underline{\mathbf{cons}(\mathsf{s}(\mathbf{0}), \texttt{inf}(\mathsf{s}(\mathsf{s}(\mathbf{0}))))}) &
\end{array}
$$

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Evaluating TRSs

Example (TRS)

$$\mathtt{inf}(y) \to \mathbf{cons}(y, \mathtt{inf}(\mathrm{s}(y)))$$

Example (Evaluation)

	$\mathtt{inf}(\mathbf{0})$	$[y/\mathbf{0}]$
$\to$	$\mathbf{cons}(\mathbf{0}, \mathtt{inf}(\mathrm{s}(\mathbf{0})))$	$[y/\mathrm{s}(\mathbf{0})]$
$\to$	$\mathbf{cons}(\mathbf{0}, \mathbf{cons}(\mathrm{s}(\mathbf{0}), \underline{\mathtt{inf}(\mathrm{s}(\mathrm{s}(\mathbf{0})))}))$	$[y/\mathrm{s}(\mathrm{s}(\mathbf{0}))]$

Evaluating TRSs

Example (TRS)

$$\mathtt{inf}(y) \to \mathbf{cons}(y, \mathtt{inf}(\mathrm{s}(y)))$$

Example (Evaluation)

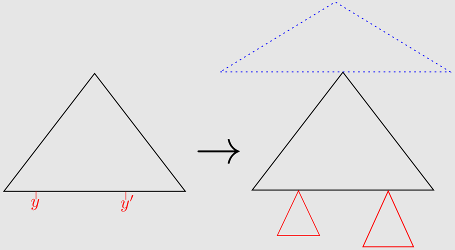
$$\begin{array}{ll|l} & \mathtt{inf}(\mathbf{0}) & [y/\mathbf{0}] \\ \to & \mathbf{cons}(\mathbf{0}, \mathtt{inf}(\mathrm{s}(\mathbf{0}))) & [y/\mathrm{s}(\mathbf{0})] \\ \to & \mathbf{cons}(\mathbf{0}, \mathbf{cons}(\mathrm{s}(\mathbf{0}), \underline{\mathtt{inf}(\mathrm{s}(\mathrm{s}(\mathbf{0})))})) & [y/\mathrm{s}(\mathrm{s}(\mathbf{0}))] \\ \to & \ldots & \end{array}$$

Evaluating TRSs

Example (TRS)

$\mathtt{inf}(y) \longrightarrow \mathbf{cons}(y, \mathtt{inf}(\mathbf{s}(y)))$

Loop

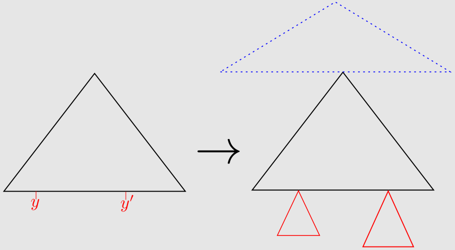


Evaluating TRSs

Example (TRS)

$\texttt{inf}(y) \to \mathbf{cons}(y, \texttt{inf}(\mathbf{s}(y)))$

Loop $\Longrightarrow$ Non-Termination

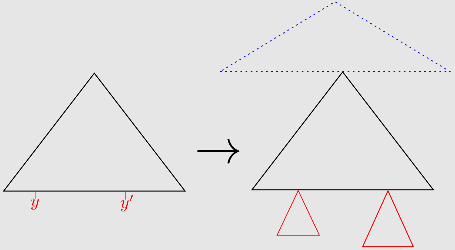


Evaluating TRSs

Example (TRS)

$\mathtt{inf}(y) \rightarrow \mathbf{cons}(y, \mathtt{inf}(\mathbf{s}(\mathbf{0})))$

Loop $\Longrightarrow$ Non-Termination



Decreasing Loops

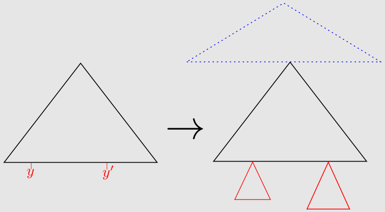
Example

$$\texttt{length}(\mathbf{cons}(x, x')) \to \mathbf{s}(\texttt{length}(x'))$$

Example

$\mathtt{length}(\mathbf{cons}(x, x')) \to \mathtt{s}(\mathtt{length}(x'))$

Loop

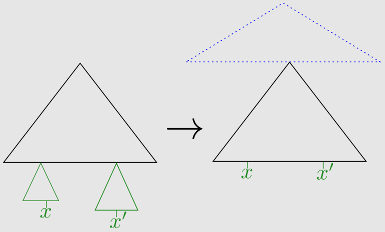


Decreasing Loops

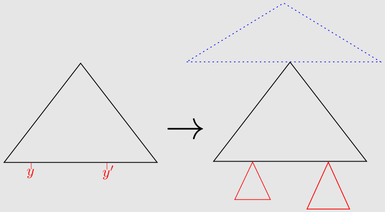
Example

$\mathtt{length}(\mathbf{cons}(x, x')) \to \mathbf{s}(\mathtt{length}(x'))$

Decreasing Loop



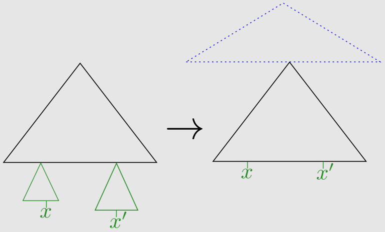
Loop



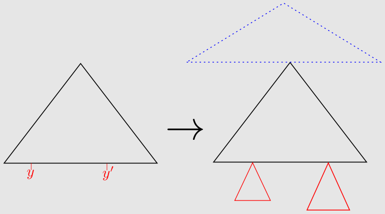
Example

$$\texttt{length}(\mathbf{cons}(x, x')) \to \mathbf{s}(\texttt{length}(x'))$$

Decreasing Loop $\Longrightarrow$ Linear Lower Bound



Loop

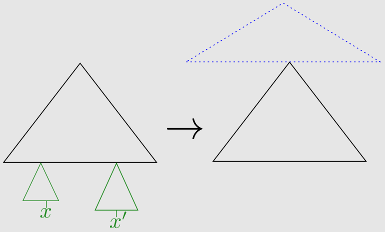


Decreasing Loops

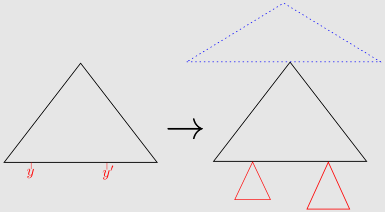
Example

$\mathtt{length}(\mathbf{cons}(x, x')) \to \mathbf{s}(\mathtt{length}(\mathbf{nil}))$

Decreasing Loop $\not\Rightarrow$ Linear Lower Bound



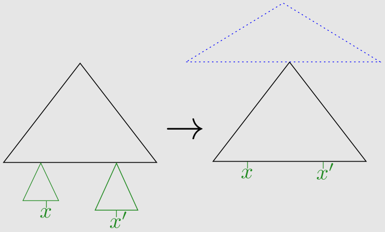
Loop



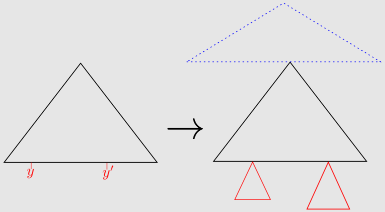
Example

$\mathtt{length}(\mathbf{cons}(x, x')) \to \mathbf{s}(\mathtt{length}(x'))$

Decreasing Loop $\Longrightarrow$ Linear Lower Bound



Loop



Decreasing Loops

Example

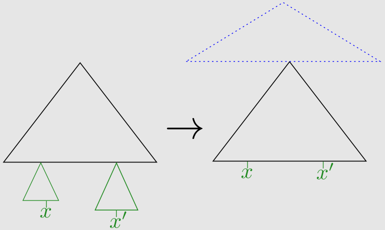
$$\begin{array}{rcl} \texttt{length}(\mathbf{cons}(x, x')) & \to & \texttt{s}(\texttt{length}(x')) \\ \curvearrowright \quad \texttt{length}(\mathbf{cons}(x, x'), y) & \to & \texttt{length}(x', \texttt{s}(y)) \end{array}$$

Example

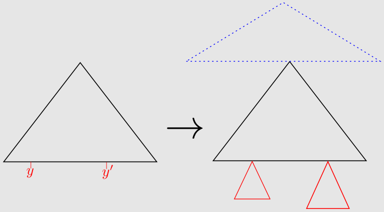
$$\mathtt{length}(\mathbf{cons}(x, x')) \to \mathtt{s}(\mathtt{length}(x'))$$

$$\curvearrowright \; \mathtt{length}(\mathbf{cons}(x, x'), y) \to \mathtt{length}(x', \mathbf{s}(y))$$

Decreasing Loops



Loops

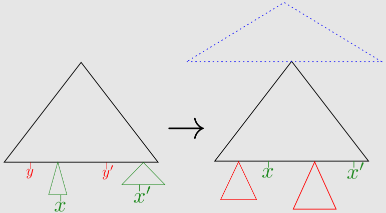


Decreasing Loops

Example

$$\begin{array}{rlcl} & \mathtt{length}(\mathbf{cons}(x, x')) & \to & \mathtt{s}(\mathtt{length}(x')) \\ \curvearrowright & \mathtt{length}(\mathbf{cons}(x, x'), y) & \to & \mathtt{length}(x', \mathbf{s}(y)) \end{array}$$

Decreasing Loops Revised



Decreasing Loops – Results

Theorem

If a TRS has a decreasing loop, then its complexity is at least linear.

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Example (Compatible Decreasing Loops)

$$\mathtt{fib}(\mathtt{s}(\mathtt{s}(x))) \to \mathtt{plus}(\mathtt{fib}(\mathtt{s}(x)), \mathtt{fib}(x))$$

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Theorem

Linear lower bounds are not semi-decidable.

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Lemma

Decreasing loops are incomplete.

Example (Compatible Decreasing Loops)

$\mathtt{fib}(\mathtt{s}(\mathtt{s}(x))) \to \mathtt{plus}(\mathtt{fib}(\mathtt{s}(x)), \mathtt{fib}(x))$

Lower Bounds for TRSs – Wrap-Up

first technique to infer worst case lower bounds for TRSs

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- key idea: generalize *loops*

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AProVE			
$\Omega(1)$	$\Omega(n)$	EXP	∞
66	600	143	90

Outline

1. Introduction
2. Lower Bounds for ITSs
3. Lower Bounds for TRSs
4. **Constant Upper Bounds for TRSs**

Narrowing

Example (from TPDB)

$$\begin{aligned} \mathbf{f}(\mathbf{a}) &\to \mathbf{g}(\mathbf{h}(\mathbf{a})) \\ \mathbf{h}(\mathbf{g}(x)) &\to \mathbf{g}(\mathbf{h}(\mathbf{f}(x))) \end{aligned}$$

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Just like rewriting, but with *unification* instead of matching.

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The complexity of a TRS is constant iff narrowing terminates for all terms $\mathtt{f}(x_1, \ldots, x_n)$.

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Lemma

Constant complexity is semi-decidable.

Constant Upper Bounds – Wrap-Up

semi-decision procedure for constant bounds

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Evaluation (TPDB – 899 examples)

- succeeds for all 57 TRSs with constant complexity

Constant Upper Bounds – Wrap-Up

semi-decision procedure for constant bounds

key idea: use *narrowing*

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Evaluation (TPDB – 899 examples)

- succeeds for all 57 TRSs with constant complexity
- solves 6 examples that couldn't be solved before

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What's next?

- integers *and* data structures

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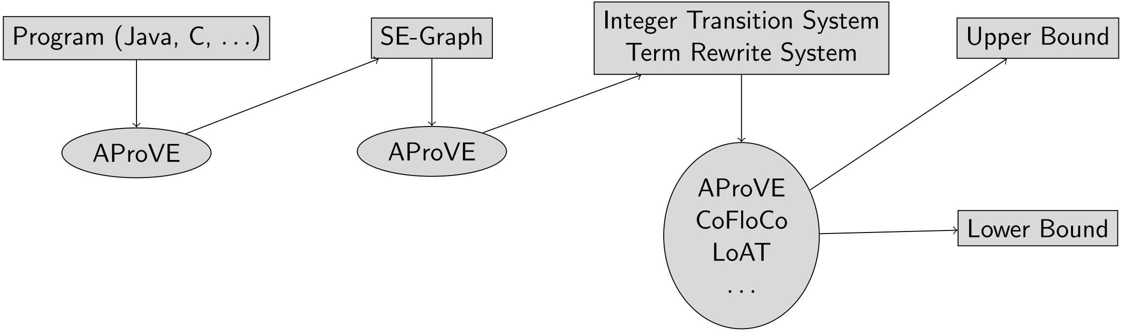

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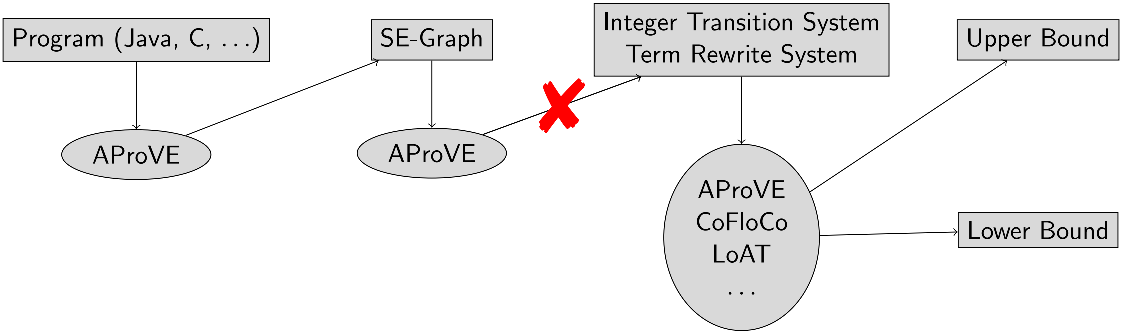

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What else?

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'16	Lower Bounds for ITSs	IJCAR
'17	Decreasing Loops for TRSs	JAR
'17		LPAR
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'17	Upper Bounds for Java	iFM
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Termination and Complexity Analysis for C — IJCAR '14, SEFM '16, JAR '17, JLAMP '18

AProVE Tool Papers — IJCAR '14, JAR '17

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Thank you!

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AProVE at TermComp

$$\frac{\text{points of AProVE}}{\text{points of best competitor}}$$

ITS	2016	2017	2018
w/ lower bounds	2.09	N/A	2.60
w/o lower bounds	0.66	N/A	0.92

TRS	2014	2015	2016	2017	2018
innermost	1.05	2.73	1.18	N/A	1.79
full	–	2.94	1.23	N/A	1.70

Recurrence Solving

Example (Recurrence Solving)

$$\begin{aligned} y^{(v+1)} &= 1 - y^{(v)} \\ x^{(v+1)} &= x^{(v)} + 1 - 3y^{(v)} \end{aligned}$$

$$\begin{aligned} y^{(v)} &= \frac{1}{2} - \frac{1}{2}(-1)^v + (-1)^v y \\ x^{(v)} &= \frac{3}{4} - \frac{3}{4}(-1)^v - \frac{3}{2}y - \frac{1}{2}v + \frac{3}{2}(-1)^v y + x \end{aligned}$$

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