Loop Detection for Lower Runtime Bounds

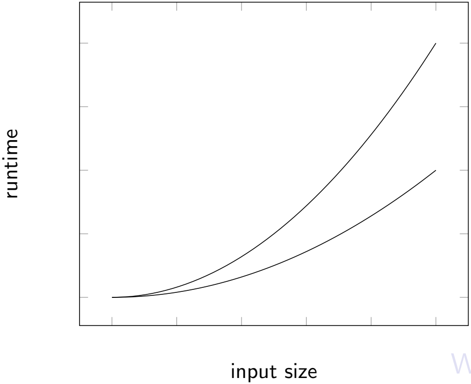
Florian Frohn [1] Jürgen Giesl [1] Jera Hensel [1]
Cornelius Aschermann [2] Thomas Ströder [1]

[1]RWTH Aachen University, Germany

[2]Ruhr University Bochum, Germany

October 20, 2016

Lower Bounds?



- worst case upper bounds
- best case lower bounds
- worst case lower bounds

Why?

- *tight* bounds
- *DoS* attacks
- *side-channel* attacks

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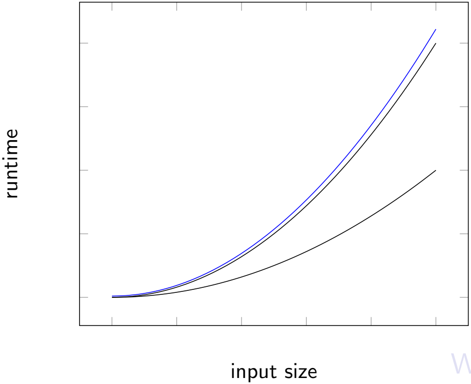

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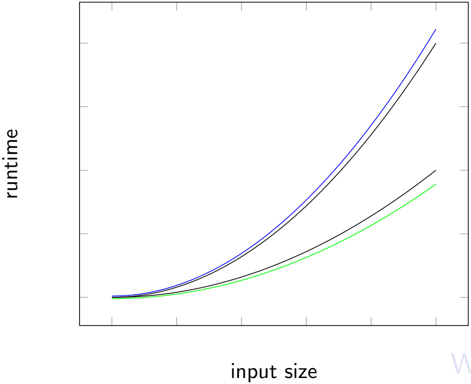

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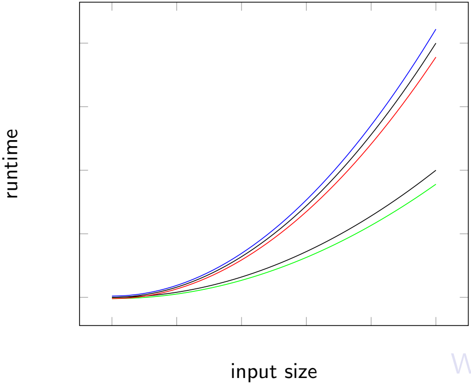

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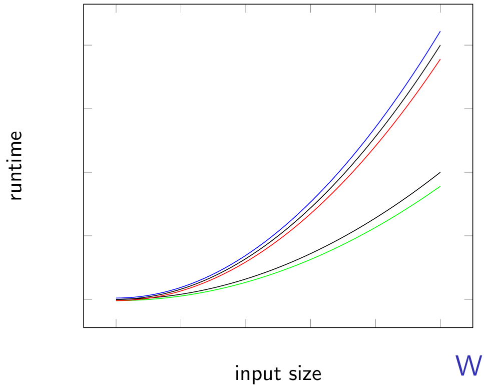

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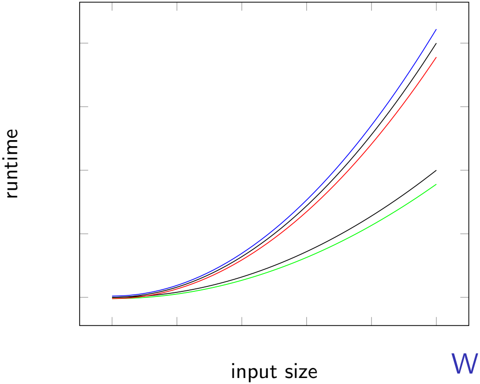

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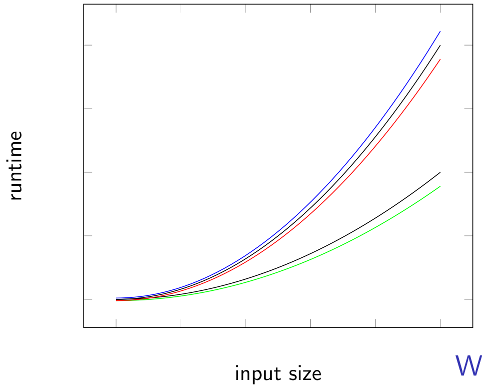

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What I'll present today...

Generalizing *Loops* to prove...

- linear and
- exponential

lower bounds for rc(n).

A First Example...

$$\begin{aligned} il(s(x), ys) &\rightarrow il(x, cons(x, ys)) \\ il(0, ys) &\rightarrow ys \end{aligned}$$

rc(n): Length of longest derivation starting with a basic term of size $m \leq n$

Basic Terms

- $il(s(0), cons(x, ys))$ ✓
- $il(x, il(0, ys))$ ✘

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$$il(x, ys) \rightarrow il(s(x), cons(x, ys)) \rightarrow il(s(s(x)), cons(s(x), cons(x, ys))) \rightarrow \dots$$

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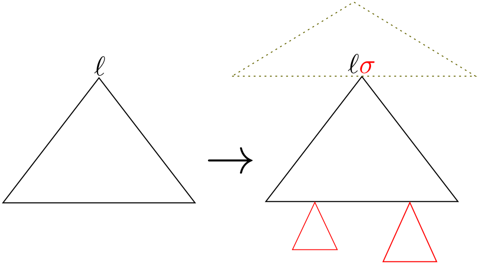

Generalizing Loops

$il(s(x), ys) \rightarrow il(x, cons(x, ys))$

$$\begin{array}{ccc} & il(x, ys) & \\ \swarrow_{\{x/s(x)\}} & & \searrow_{\{ys/cons(x,ys)\}} \\ il(s(x), ys) & \rightarrow & il(x, cons(x, ys)) \end{array}$$

Generalizing Loops

$il(s(x),ys) \to il(x,cons(x,ys))$

$$\begin{tikzcd}[column sep=small]
& il({\color{blue}x}, {\color{red}ys}) \arrow[dl, dashed, "{\color{blue}\{x/s(x)\}}"'] \arrow[dr, dashed, "{\color{red}\{ys/cons(x,ys)\}}"] & \\
il({\color{blue}s(x)}, ys) \arrow[rr, phantom, "\to"] & & il(x, {\color{red}cons(x, ys)})
\end{tikzcd}$$

θ: Pumping Substitution σ: Result Substitution $\bar{\ell}$: Base Term

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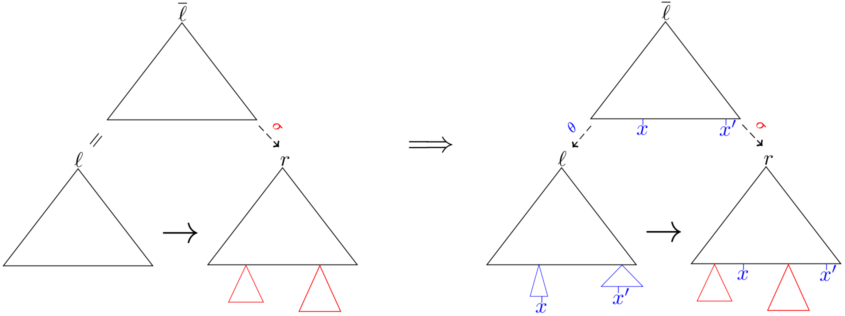

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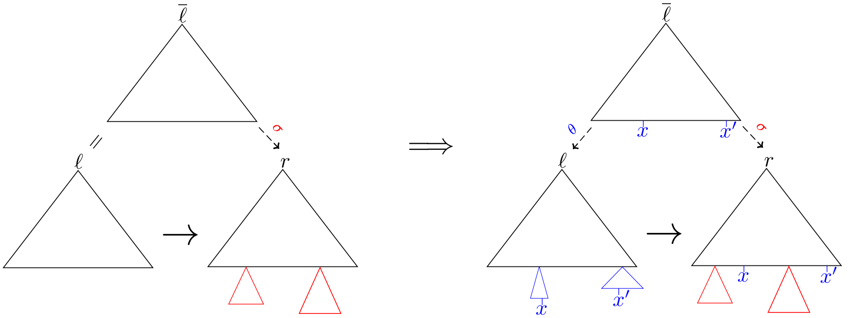
$\{x/s(x)\}$ $\quad$ $\{ys/cons(x, ys)\}$



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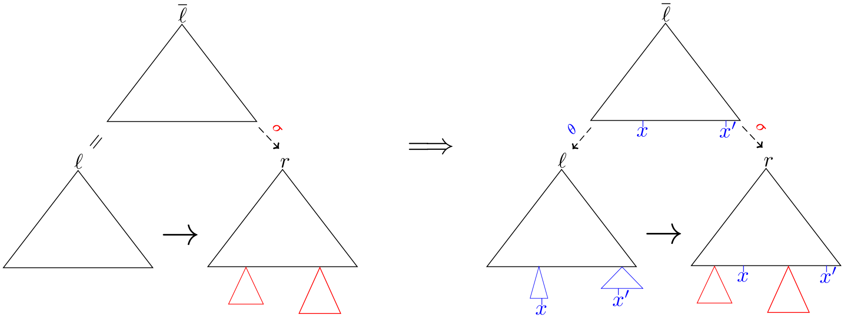

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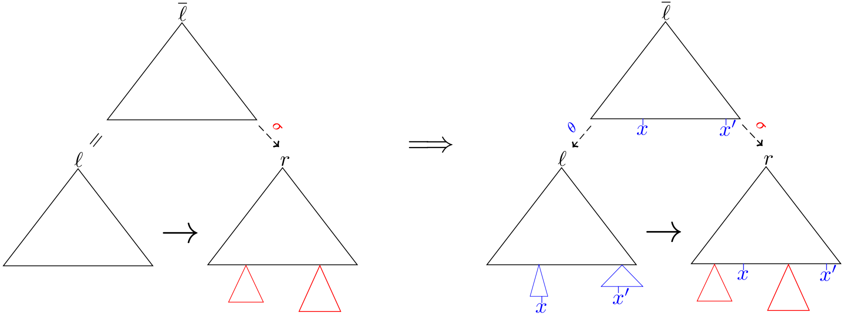

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$\ell\theta^n = il(s^{n+1}(x),ys) \to il(s^n(x),cons(s^n(x),ys)) \to \ldots$

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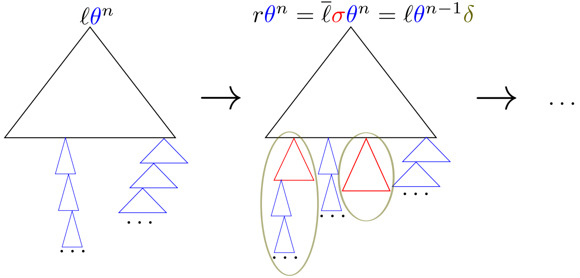
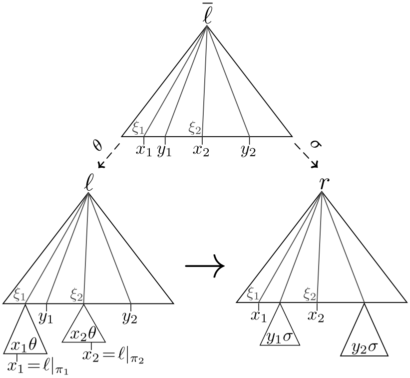




$f(n(x, x)) \rightarrow f(x)$

Definition (Decreasing Loop)

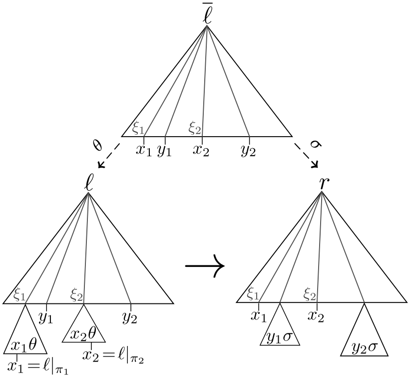
$\ell \rightarrow^+ C[r]$ is a *decreasing loop* if there are variables $x_1, \ldots, x_m$ and positions $\pi_1, \ldots, \pi_m$ s.t.:

- ℓ linear and basic
- $\ell|_{\pi_i} = x_i$
- $r|_{\xi_i} = x_i$ for some $\xi_i < \pi_i$
- $\bar{\ell}$ matches r

Theorem

If a TRS has a decreasing loop, then $\mathrm{rc}(n) \in \Omega(n)$.

Decreasing Loops



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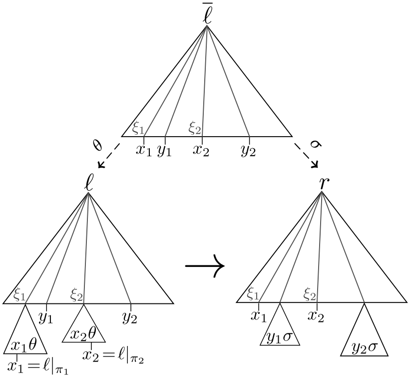

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 & & \mathit{fib}(s^{n-2}(0)) \ \ldots \\
 & \mathit{fib}(s^{n-1}(0)) \rightarrow & \\
 & & \mathit{fib}(s^{n-3}(0)) \ \ldots \\
\mathit{fib}(s^{n}(0)) \rightarrow & & \\
 & & \mathit{fib}(s^{n-3}(0)) \ \ldots \\
 & \mathit{fib}(s^{n-2}(0)) \rightarrow & \\
 & & \mathit{fib}(s^{n-4}(0)) \ \ldots
\end{array}
$$

Exponential Bounds

$$fib(s(s(x))) \to p(fib(s(x)), fib(x))$$

$$\begin{tikzcd}[column sep=small] & \overline{\ell} = fib(s(x)) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\emptyset"] & \\ fib(s(s(x))) & \to & C[fib(s(x))] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overline{\ell}' = fib(x) \arrow[dl, dashed, "\theta'"'] \arrow[dr, dashed, "\emptyset"] & \\ fib(s(s(x))) & \to & C'[fib(x)] \end{tikzcd}$$

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\begin{tikzcd}
 & & tr(n^{m-2}(l,l)) \dots \\
 & tr(n^{m-1}(l,l)) \arrow[ur] \arrow[r] & tr(n^{m-2}(l,l)) \dots \\
tr(n^{m}(l,l)) \arrow[ur] \arrow[dr] & & \\
 & tr(n^{m-1}(l,l)) \arrow[r] \arrow[dr] & tr(n^{m-2}(l,l)) \dots \\
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$$\begin{array}{ccc} & \overline{\ell} = tr(x) & \\ \swarrow_{\theta} & & \searrow_{\emptyset} \\ tr(n(x,y)) & \to & C[tr(x)] \end{array}$$

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$$\overline{\ell} = tr(x)$$

$$tr(n(x,y)) \xleftarrow{\theta} \cdots \qquad tr(n(x,y)) \to C[tr(x)], \quad \overline{\ell} \xrightarrow{\emptyset} C[tr(x)]$$

$$\overline{\ell}' = tr(y)$$

$$tr(n(x,y)) \to C'[tr(y)], \quad \overline{\ell}' \xrightarrow{\theta'} tr(n(x,y)), \quad \overline{\ell}' \xrightarrow{\emptyset} C'[tr(y)]$$

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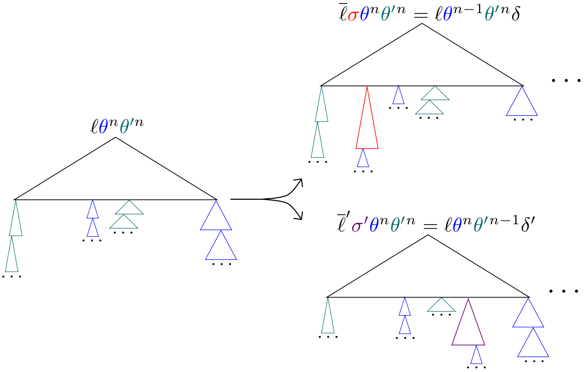
Multiple Decreasing Loops



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Commutativity

$$\theta\theta' \stackrel{?}{=} \theta'\theta$$

$$\begin{tikzcd} & \overline{\ell} = tr(x) \arrow[dl, dashed, "\theta"'] \arrow[dr, dashed, "\emptyset"] & \\ tr(n(x,y)) \arrow[rr] & & C[tr(x)] \end{tikzcd}$$

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Compatible Decreasing Loops

Definition

Two decreasing loops are *compatible* iff

- σ and σ' don't interfere with θ and θ'
- $\theta\,\theta' = \theta'\theta$

Theorem

If a TRS has b compatible decreasing loops, then $\mathrm{rc}(n) \in \Omega(b^n)$.

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Experiments (865 Examples)

AProVE without Decreasing Loops

rc(n)	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	$\Omega(2^n)$	$\Omega(3^n)$	$\Omega(\omega)$
Σ	192	572	73	14	1	12	1	–

AProVE with Decreasing Loops

rc(n)	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	$\Omega(2^n)$	$\Omega(3^n)$	$\Omega(\omega)$
Σ	29	533	56	11	1	144	1	90

Conclusion

- Generalized Loops to prove linear lower bounds
- Generalized Loops to prove exponential lower bounds
- Experimental results $\rightarrow$ applicable to almost all TRSs from TPDB

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- Generalized Loops to prove linear lower bounds
- Generalized Loops to prove exponential lower bounds
- Experimental results $\rightarrow$ applicable to almost all TRSs from TPDB

Decidability

Consider the class of linear basic TRSs.

$$\begin{aligned} & \mathrm{rc}(n) \in \Omega(n) \\ \iff & \mathrm{rc}(n) \notin \mathcal{O}(1) \\ \iff & \text{narrowing basic terms does not terminate} \\ \iff & \text{rewriting } \textit{infinite} \text{ basic terms does not terminate} \end{aligned}$$

Let $\mathcal{R}_{\mathcal{M}}$ be the TRS encoding the Turing machine $\mathcal{M}$.

$$\text{rewriting infinite basic terms does not terminate} \iff \mathcal{M} \text{ is immortal}$$

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Conclusion

- Generalized Loops to prove linear lower bounds
- Generalized Loops to prove exponential lower bounds
- Experimental results $\rightarrow$ applicable to almost all TRSs from TPDB
- Decidability of $\mathrm{rc}(n) \in \Omega(n)$

Experiments (865 Examples)

Without Decreasing Loops

$\mathrm{rc}_{\mathcal{R}}(n)$	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	$\Omega(2^n)$	$\Omega(3^n)$	$\Omega(\omega)$
$\mathcal{O}(1)$	(34)	–	–	–	–	–	–	–
$\mathcal{O}(n)$	41	114	–	–	–	–	–	–
$\mathcal{O}(n^2)$	5	10	3	–	–	–	–	–
$\mathcal{O}(n^3)$	1	1	1	1	–	–	–	–
$\mathcal{O}(n^{>3})$	–	2	–	–	–	–	–	–
$\mathcal{O}(2^n)$	–	–	–	–	–	–	–	–
$\mathcal{O}(3^n)$	–	–	–	–	–	–	–	–
$\mathcal{O}(\omega)$	145	445	69	13	1	12	1	–

With Decreasing Loops

$\mathrm{rc}_{\mathcal{R}}(n)$	$\Omega(1)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	$\Omega(n^{>3})$	$\Omega(2^n)$	$\Omega(3^n)$	$\Omega(\omega)$
$\mathcal{O}(1)$	(34)	–	–	–	–	–	–	–
$\mathcal{O}(n)$	15	140	–	–	–	–	–	–
$\mathcal{O}(n^2)$	–	15	3	–	–	–	–	–
$\mathcal{O}(n^3)$	–	2	1	1	–	–	–	–
$\mathcal{O}(n^{>3})$	–	2	–	–	–	–	–	–
$\mathcal{O}(2^n)$	–	–	–	–	–	–	–	–
$\mathcal{O}(3^n)$	–	–	–	–	–	–	–	–
$\mathcal{O}(\omega)$	14	374	52	10	1	144	1	90