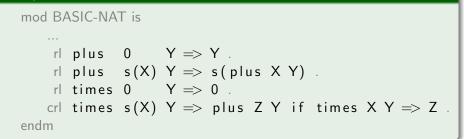
# Analyzing Runtime Complexity via Innermost Runtime Complexity

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May 11, 2017



$$\begin{array}{rcl} \mathsf{plus}(0,y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x),y) & \to & \mathsf{s}(\mathsf{plus}(x,y)) \\ \mathsf{times}(0,y) & \to & 0 \\ \mathsf{times}(\mathsf{s}(x),y) & \to & \mathsf{plus}(z,y) & \Leftarrow & \mathsf{times}(x,y) \approx z \end{array}$$

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Goal: Prove upper bound on worst case complexity

Transformation by Cynthia Kop, Aart Middeldorp, and Thomas Sternagel

"Complexity of Conditional Term Rewriting", LMCS '17

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#### Leading Example $\mathcal{R}_{times}$

$plus(0, y, \top, x_2)$	$\rightarrow$	у
$plus(s(x), y, x_1, \top)$	$\rightarrow$	$s(plus(x, y, \top, \top))$
$times(0, y, \top, x_2)$	$\rightarrow$	0
times(s(x), y, x <sub>1</sub> , $\top$ )	$\rightarrow$	$times_2^1(s(x), y, x_1, times(x, y, \top, \top))$
times <sup>1</sup> <sub>2</sub> (s(x), y, x <sub>1</sub> , z)	$\rightarrow$	$plus(z, y, \top, \top)$

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Let's analyze it using leading tools!

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$\operatorname{times}_{2}^{1}(\operatorname{s}(x), y, x_{1}, z)$	$\rightarrow$	$plus(z, y, \top, \top)$

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- AProVE: full rewriting not supported

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# Outline

#### Preliminaries

- rc and irc
- NDG Rewriting

2 Handling Constructor Systems

3 Handling Non-Constructor Systems



#### • rc maps $n \in \mathbb{N}$ to the length of the longest rewrite sequence s.t.

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#### Example

• **plus**(0, s(0))

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- $plus(0, plus(0, s(0))) \times$

- rc maps  $n \in \mathbb{N}$  to the length of the longest rewrite sequence s.t. (A) size of start term bounded by n
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- irc: similar, but just considers innermost sequences

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"Generalized innermost rewriting" (RTA '05)

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Leading Example $\mathcal{R}_{times}$					
plus(0, y) plus(s(x), y) times(0, y) times(s(x), y)	${\rightarrow}$	$s(\mathbf{plus}(x, y))$			

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Leading Exam	ple	$\mathcal{R}_{times}$	
times(0, y)	$\stackrel{\rightarrow}{\rightarrow}$	$s(\mathbf{plus}(x, y))$	

#### Example

• times(s(0), plus(0,0))  $\rightarrow plus(times(0, plus(0,0)), plus(0,0))$ 

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rc maps n to the length of the longest rewrite sequence s.t.

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- Goal: Prove that all sequences starting with basic terms are ndg Use irc techniques to analyze rc

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# 2 Handling Constructor Systems

- 3 Handling Non-Constructor Systems
- 4 Experimental Results, Conclusion

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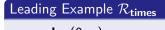
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$$\frac{\mathsf{plus}(0,y)}{\mathsf{Reminder}} \rightarrow$$

- tim no duplication of defined symbols
  - $\sim$  innermost rewriting is worst

$$\sim$$
 rc = irc

۲

- $\curvearrowright$  irc techniques applicable for rc
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•  $plus(x, \Box)$  does not match  $plus(s(\Box), y)$ 



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```
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**Goal:** compute sets of contexts *Dup* and *Def* 



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- **Goal:** compute sets of contexts Dup and DefDup and Def don't overlap  $\sim$  rc = irc



C matches D if

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### Overlapping contexts

C and D overlap if both match some E

## Example

•  $plus(\Box, y)$  matches  $plus(s(\Box), y)$ 

 Intuition: plus(□, y) represents "marked" terms plus(times(x, z), y), plus(s(times(0, 0)), 0), ...

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Example

collect left-hand sides of rules with non-linear right-hand sides

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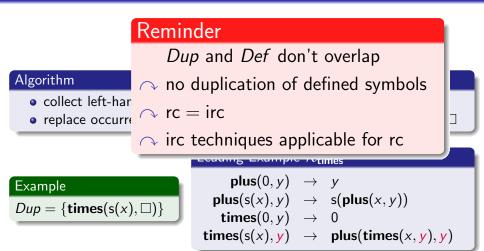
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Examp	le

## Leading Example $\mathcal{R}_{times}$

# The easy one: Computing Dup



## Initialization

## Example

## Leading Example $\mathcal{R}_{\mathsf{times}}$

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#### Initialization

• collect right-hand sides with nested defined symbols

#### Example

## Leading Example $\mathcal{R}_{times}$

$$\begin{array}{rcl} \mathsf{plus}(0,y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x),y) & \to & \mathsf{s}(\mathsf{plus}(x,y)) \\ \mathsf{times}(0,y) & \to & 0 \\ \mathsf{times}(\mathsf{s}(x),y) & \to & \mathsf{plus}(\mathsf{times}(x,y),y) \end{array}$$

• collect right-hand sides with nested defined symbols

#### Example

$$plus(0, y) \rightarrow y$$
  

$$plus(s(x), y) \rightarrow s(plus(x, y))$$
  

$$times(0, y) \rightarrow 0$$
  

$$times(s(x), y) \rightarrow plus(times(x, y), y)$$

- collect right-hand sides with nested defined symbols
- replace nested defined symbols with  $\Box \curvearrowright {\cal C}$

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Example

 $\mathsf{plus}(\Box, y)$ 

$$\begin{array}{rcl} \mathsf{plus}(0,y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x),y) & \to & \mathsf{s}(\mathsf{plus}(x,y)) \\ \mathsf{times}(0,y) & \to & 0 \\ \mathsf{imes}(\mathsf{s}(x),y) & \to & \mathsf{plus}(\mathsf{times}(x,y),y) \end{array}$$

- collect right-hand sides with nested defined symbols
- replace nested defined symbols with  $\Box \curvearrowright C$
- add  $\lfloor C \rfloor$  to *Def*

Leading Example  $\mathcal{R}_{times}$ 

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Leading Example  $\mathcal{R}_{times}$ 

Example

 $Def = \{ \lfloor \mathsf{plus}(\Box, y) \rfloor \}$ 

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Leading Example  $\mathcal{R}_{times}$ 

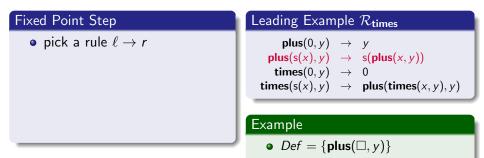
Example

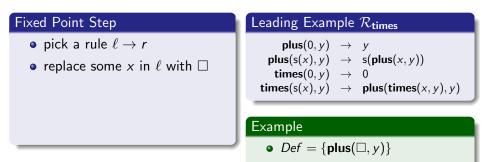
 $Def = \{ \mathsf{plus}(\Box, y) \}$ 

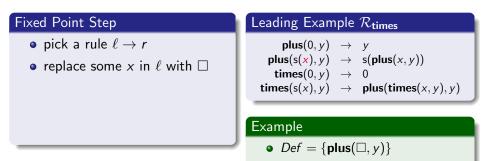
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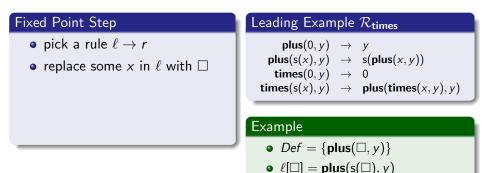
Fixed Point Step	Leading Example $\mathcal{R}_{times}$
	$\begin{array}{rcl} & plus(0,y) & \to & y \\ & plus(s(x),y) & \to & s(plus(x,y)) \\ & times(0,y) & \to & 0 \\ & times(s(x),y) & \to & plus(times(x,y),y) \end{array}$
	Example • $Def = \{ plus(\Box, y) \}$

Fixed Point Step	Leading Example $\mathcal{R}_{times}$
• pick a rule $\ell \to r$	$\begin{array}{rcl} & \textbf{plus}(0,y) & \rightarrow & y \\ & \textbf{plus}(s(x),y) & \rightarrow & \textbf{s}(\textbf{plus}(x,y)) \\ & \textbf{times}(0,y) & \rightarrow & 0 \\ & \textbf{times}(\textbf{s}(x),y) & \rightarrow & \textbf{plus}(\textbf{times}(x,y),y) \end{array}$
	Example • $Def = \{ plus(\Box, y) \}$









# Fixed Point Step

- pick a rule  $\ell \rightarrow r$
- replace some x in  $\ell$  with  $\Box$
- if  $\ell[\Box]$  overlaps with  $D \in Def$

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• 
$$Def = {$$
**plus** $(\Box, y)$ }

• 
$$\ell[\Box] = \mathsf{plus}(\mathsf{s}(\Box), y)$$

# Fixed Point Step

- pick a rule  $\ell \rightarrow r$
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• 
$$Def = { plus(\Box, y), \lfloor plus(\Box, y) \rfloor }$$

• 
$$\ell[\Box] = \mathsf{plus}(\mathsf{s}(\Box), y)$$

# Fixed Point Step

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# Fixed Point Step• pick a rule $\ell \to r$ • replace some x in $\ell$ with $\Box$ • if $\ell[\Box]$ overlaps with $D \in Def$ • pick a subterm f(...x...) of r• add $\lfloor f(...\Box...) \rfloor$ to Def• $Leading Example <math>\mathcal{R}_{times}$ • plus $(0, y) \to y$ • plus $(s(x), y) \to s(plus(x, y))$ • times $(0, y) \to 0$ • times $(s(x), y) \to 0$ • Def• $Leading Example <math>\mathcal{R}_{times}$ • Def• $Def = \{plus(\Box, y)\}$ • $\ell[\Box] = plus(s(\Box), y)$

 $Dup = \{ times(s(x), \Box) \}$  and  $Def = \{ plus(\Box, y) \}$  don't overlap  $\frown$  rc = irc!

# Outline

# Preliminaries

- rc and irc
- NDG Rewriting
- 2 Handling Constructor Systems
- 3 Handling Non-Constructor Systems
  - 4 Experimental Results, Conclusion

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  - nested defined symbols only below **plus**'s first argument

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- $\land$  **plus**(*x*, **plus**(*y*, *z*)) not reachable from basic terms!

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 ∧ plus(x, plus(y, z)) not reachable from basic terms!

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- nested defined symbols only below plus's first argument
- $\frown$  **plus**(*x*, **plus**(*y*, *z*)) not reachable from basic terms!
  - information which defined symbols can be nested often crucial
- $\sim$  similar fixed point algorithm

TcT	AProVE	TcT preproc	AProVE & TcT
209	270	299	308

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• powerful sufficient criterion for rc = irc

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- powerful sufficient criterion for rc = irc
- easy to automate
- $\curvearrowright\,$  future irc techniques applicable for rc

TcT	AProVE	TcT preproc	AProVE & TcT	AProVE++
209	270	299	308	324

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TcT	AProVE	TcT preproc	AProVE & TcT	AProVE++
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- powerful sufficient criterion for rc = irc
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- $\sim$  future irc techniques applicable for rc
  - significant improvement of the state of the art